Final Exam Mathematical Physics, Prof. G. Palasantzas

- Total number of points 100
- 10 points for coming to the final exam
- Justify briefly your answers for all problems

<u>Problem 1 (10 points)</u> Is the infinite series $\sum_{n=1}^{\infty}\left(1+\frac{2}{n}\right)^{4n}$ convergent?

<u>Problem 2 (10 points)</u> Consider the series $\sum_{n=1}^{\infty}\frac{(x+3)^n}{n5^n}$. For which x is the series convergent?

<u>Problem 3 (15 points)</u>
(a: 8 points)
(b: 7 points)

Let L be a nonzero real number.

(a) Show that the boundary-value problem $y'' + \lambda y = 0$, $y(0) = 0$, $y(L) = 0$ has only the trivial solution $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) For the case $\lambda > 0$, find the values of λ for which this problem has a nontrivial solution and give the corresponding solution.

<u>Problem 4 (15 points)</u>

Find periodic solutions of the form $X(t) = \sum_{n=-\infty}^{+\infty} X_n e^{int}$ for the second order differential equation:

$$m\frac{d^2X}{dt^2} + c\frac{dX}{dt} + kX = F_1\cos(2t) + F_2\cos(4t)$$

where *m, c*, and *k* are real positive numbers.

Problem 5 (20 points)

(a: 10 points) Consider the periodic function $F(x) = \sum_{n=-\infty}^{+\infty} f_n e^{inx}$ in the interval [-π, π].

Prove that: $\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(x)|^2 dx = \sum_{n=-\infty}^{+\infty} |f_n|^2$

(b: 10 points) Consider the Fourier transform definitions $F(k) = \int_{-\infty}^{+\infty} F(x)e^{-i2\pi kx}dx$ and $\delta(k) = \int_{-\infty}^{+\infty} e^{-i2\pi kx}dx$

Calculate the Fourier transform of $F(x) = 5 + 3cos^2(5\pi\varepsilon x)$

Problem 6 (20 points)

(a: 10 points) Consider the boundary value problem for the one-dimensional heat equation of a bar with length L and isolated temperature ends (Neumann boundary conditions):

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad u = u(x,t), \quad t > 0, \quad 0 < x < L$$

$$u(x,0) = f(x),$$

$$u_x(x=0,t) = u_x(x=L,t) = 0 \ (u_x = \vartheta u/\vartheta x \text{ , t≥0}).$$

Show that the general solution is given in this case by: $u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \cos \mu_n x$

where $\lambda_n = c\mu_n$ and $\mu_n = n\pi/L$.

(b: 10 points) Calculate the solution u(x, t) for *f(x)=D+2Acos(3πx/L)* cos^2*(6πx/L)*

Problem 1

Take term $a_n = \left(1 + \frac{2}{n}\right)^{4n}$

Take $W_n = \ln a_n = 4n \ln\left(1 + \frac{2}{n}\right)$

$$\lim_{n\to\infty} W_n = 4 \lim_{n\to\infty} \frac{\ln\left(1+\frac{2}{n}\right)}{\frac{1}{n}}$$

Apply l'Hospitals rule

$$\lim_{n\to\infty} W_n = 4 \lim_{n\to\infty} \frac{\left[1/1+\frac{2}{n}\right]\left(-\frac{2}{n^2}\right)}{-1/n^2} \Rightarrow$$

$$\lim_{n\to\infty} W_n = 8 \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 8$$

Therefore $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{W_n} = e^{\lim_{n\to\infty} W_n} = e^8 \neq 0$

Since $\lim_{n\to\infty} a_n \neq 0$ $\sum_{n=1}^{\infty} a_n$ is divergent

Problem 2

Apply ratio test

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{\frac{|x+3|^{n+1}}{(n+1)5^{n+1}}}{\frac{|x+3|^n}{n5^n}} =$$

$$= \lim_{n \to \infty} \frac{|x+3|}{(1+\frac{1}{n})5} = \frac{|x+3|}{5} < 1 \iff$$

$$|x+3| < 5 \iff -8 < x < 2$$

- $\sum_{n=1}^{\infty} a_n$ obsolusely convergent for $-8 < x < 2$
- For $x = 2$ we get the series $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent Harmonic series
- For $x = -8$ we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which is an olternating series conditionally convergent

Thus $\sum_{n=1}^{\infty} a_n$ is convergent for $-8 \leq x < 2$

Problem 3

(a) *Case 1* ($\lambda = 0$): $y'' + \lambda y = 0 \quad \Rightarrow \quad y'' = 0$ which has an auxiliary equation $r^2 = 0 \quad \Rightarrow \quad r = 0 \quad \Rightarrow \quad y = c_1 + c_2 x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 L \quad \Rightarrow \quad c_1 = c_2 = 0$. Thus $y = 0$.

Case 2 ($\lambda < 0$): $y'' + \lambda y = 0$ has auxiliary equation $r^2 = -\lambda \quad \Rightarrow \quad r = \pm\sqrt{-\lambda}$ [distinct and real since $\lambda < 0$] $\Rightarrow$ $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $y(0) = 0$ and $y(L) = 0$. Thus $0 = y(0) = c_1 + c_2$ $(*)$ and $0 = y(L) = c_1 e^{\sqrt{-\lambda}L} + c_2 e^{-\sqrt{-\lambda}L}$ $(\dagger)$.

Multiplying $(*)$ by $e^{\sqrt{-\lambda}L}$ and subtracting $(\dagger)$ gives $c_2\left(e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L}\right) = 0 \quad \Rightarrow \quad c_2 = 0$ and thus $c_1 = 0$ from $(*)$.

Thus $y = 0$ for the cases $\lambda = 0$ and $\lambda < 0$.

(b) $y'' + \lambda y = 0$ has an auxiliary equation $r^2 + \lambda = 0 \quad \Rightarrow \quad r = \pm i\sqrt{\lambda} \quad \Rightarrow \quad y = c_1 \cos\sqrt{\lambda}\,x + c_2 \sin\sqrt{\lambda}\,x$ where $y(0) = 0$ and $y(L) = 0$. Thus, $0 = y(0) = c_1$ and $0 = y(L) = c_2 \sin\sqrt{\lambda}L$ since $c_1 = 0$. Since we cannot have a trivial solution, $c_2 \neq 0$ and thus $\sin\sqrt{\lambda}\,L = 0 \quad \Rightarrow \quad \sqrt{\lambda}\,L = n\pi$ where n is an integer $\Rightarrow \quad \lambda = n^2\pi^2/L^2$ and $y = c_2 \sin(n\pi x/L)$ where n is an integer.

Problem 4

write $\cos(2t) = \frac{1}{2} e^{i2t} + \frac{1}{2} e^{-i2t}$

$\cos(4t) = \frac{1}{2} e^{i4t} + \frac{1}{2} e^{-i4t}$

Thus we have

$$m x'' + c x' + k x = \overbrace{\frac{F_1}{2} e^{i2t} + \frac{F_1}{2} e^{-i2t} + \frac{F_2}{2} e^{i4t} + \frac{F_2}{2} e^{-i4t}}^{F(t)}$$

substitute $x(t) = \sum_{-\infty}^{+\infty} x_n e^{int}$ into the differential equation:

$$\sum_{-\infty}^{+\infty} (-mn^2) x_n e^{int} + \sum_{-\infty}^{+\infty} c(in) x_n e^{int} + \sum_{-\infty}^{+\infty} k x_n e^{int} = F(t)$$

$$\sum_{-\infty}^{+\infty} [(k - mn^2) + icn] x_n e^{int} = F(t)$$

olny the terms $n = \pm 2, \pm 4$ will have $x_n \neq 0$

so that:

$[(k-4m) + i2c] x_2 = \frac{F_1}{2}$, $[(k-4m) - i2c] x_{-2} = \frac{F_1}{2}$

$[(k-16m) + i4c] x_4 = \frac{F_2}{2}$, $[(k-16m) - i4c] x_{-4} = \frac{F_2}{2}$

Thus the have the periodic solution

$$x(t) = \frac{F_1 e^{i2t}}{2[(k-4m) + i2c]} + \frac{F_1 e^{-i2t}}{2[(k-4m) - i2c]} + \frac{F_2 e^{i4t}}{2[(k-16m) + i4c]} + \frac{F_2 e^{-i4t}}{2[(k-16m) - i4c]}$$

Problem 5

(a) $$\int_{-\pi}^{\pi} |F(x)|^2 dx = \int_{-\pi}^{\pi} \sum_{-\infty}^{+\infty} f_n e^{inx} \sum_{-\infty}^{+\infty} f_m^* e^{-imx} dx \quad (1)$$

$|F(x)|^2 = FF^*$, F^* complex conjugate

Thus we have

$$\int_{-\pi}^{\pi} |F(x)|^2 dx = \sum_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} f_n f_m^* \int_{-\pi}^{\pi} e^{i(n-m)x} dx \quad (2)$$

$$\int_{-\pi}^{\pi} e^{i(n-m)x} dx = \underbrace{\int_{-\pi}^{\pi} \cos[(n-m)x] dx}_{= 2\pi\delta_{mn}} + i \underbrace{\int_{-\pi}^{\pi} \sin[(n-m)x] dx}_{= 0}$$

so we have $\int_{-\pi}^{\pi} e^{i(n-m)x} dx = 2\pi \delta_{mn}$ (3)

Alternatively one can also say:

$$\int_{-\pi}^{\pi} e^{i(n-m)x} dx \rightarrow \begin{cases} 2\pi & n=m \\ \frac{1}{i(n-m)} e^{i(n-m)x}\Big|_{-\pi}^{\pi} \ (n \neq m) = 0 \end{cases} = 2\pi\delta_{mn}$$

$$\delta_{mn} \rightarrow \begin{cases} 1 & m=n \\ 0 & m \neq n \end{cases}$$

(2) & (3) $$\int_{-\pi}^{\pi} |F(x)|^2 dx = \sum_{-\infty}^{+\infty} \sum_{-\infty}^{+\infty} f_n f_m^* \, 2\pi \delta_{mn} \Rightarrow$$

$$\int_{-\pi}^{\pi} |F(x)|^2 dx = 2\pi \sum_{-\infty}^{+\infty} |f_n|^2 \Leftrightarrow$$

$$\boxed{\frac{1}{2\pi} \int_{-\pi}^{\pi} |F(x)|^2 dx = \sum_{-\infty}^{+\infty} |f_n|^2}$$

(b) $F(x) = 5 + \frac{3}{4}\left(e^{i5\pi\varepsilon x} + e^{-i5\pi\varepsilon x}\right)^2 \Rightarrow$

$$F(x) = 5 + \frac{3}{4}\left(e^{i10\pi\varepsilon x} + e^{-i10\pi\varepsilon x} + 2\right) \Rightarrow$$

$$F(x) = \frac{13}{2} + \frac{3}{4} e^{i10\pi\varepsilon x} + \frac{3}{4} e^{-i10\pi\varepsilon x}$$

calculate $F(k)$:

$$F(k) = \int_{-\infty}^{+\infty} \left[\frac{13}{2} + \frac{3}{4} e^{i10\pi\varepsilon x} + \frac{3}{4} e^{-i10\pi\varepsilon x}\right] e^{-i2\pi kx}\, dx$$

$$\Rightarrow F(k) = \frac{13}{2} \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi kx}\, dx}_{\delta(k)} + \frac{3}{4} \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi(k-5\varepsilon)x}\, dx}_{\delta(k-5\varepsilon)} + \frac{3}{4} \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi(k+5\varepsilon)x}\, dx}_{\delta(k+5\varepsilon)}$$

$$\Rightarrow \boxed{F(k) = \frac{13}{2}\delta(k) + \frac{3}{4}\delta(k-5\varepsilon) + \frac{3}{4}\delta(k+5\varepsilon)}$$

(6a) Problem 6

Assuming that $u(x,t) = X(x)T(t)$, the heat equation (1) becomes

$$XT' = c^2 X'' T.$$

This implies

$$\frac{X''}{X} = \frac{T'}{c^2 T} = k,$$

which we write as

$$X'' - kX = 0, \quad (4)$$

$$T' - c^2 kT = 0. \quad (5)$$

The initial conditions (2) become $X'(0)T(t) = X'(L)T(t) = 0$, or

$$X'(0) = X'(L) = 0. \quad (6)$$

$$k = -\mu^2 < 0$$

The ODE (4) is now $X'' + \mu^2 X = 0$ with solutions

$$X = c_1 \cos \mu x + c_2 \sin \mu x.$$

The boundary conditions (6) yield

$$0 = X'(0) = -\mu c_1 \sin 0 + \mu c_2 \cos 0 = \mu c_2,$$
$$0 = X'(L) = -\mu c_1 \sin \mu L + \mu c_2 \cos \mu L.$$

The first of these gives $c_2 = 0$. In order for X to be nontrivial, the second shows that we also need

$$\sin \mu L = 0.$$

This can occur if and only if $\mu L = n\pi$, that is

$$\mu = \mu_n = \frac{n\pi}{L}, \ n = \pm 1, \pm 2, \pm 3, \ldots$$

Choosing $c_1 = 1$ yields the solutions

$$X_n = \cos \mu_n x, \ n = 1, 2, 3, \ldots$$

For each n the corresponding equation (5) for T becomes $T' = -\lambda_n^2 T$, with $\lambda_n = c\mu_n$. Up to a constant multiple, the solution is

$$T_n = e^{-\lambda_n^2 t}.$$

Multiplying these together gives the n**th normal mode**

$$\boxed{u_n(x,t) = X_n(x)\,T_n(t) = e^{-\lambda_n^2 t}\cos \mu_n x, \ n = 1, 2, 3, \ldots}$$

where $\mu_n = n\pi/L$ and $\lambda_n = c\mu_n$.
The principle of superposition now guarantees that for any choice of constants $a_0, a_1, a_2, \ldots$

$$u(x,t) = a_0 u_0 + \sum_{n=1}^{\infty} a_n u_n = a_0 + \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t}\cos \mu_n x \qquad (7)$$

is a solution of the heat equation (1) with the Neumann boundary conditions (2).

(b) $2\cos(3\pi x/L)\cos^2(6\pi x/L)=\cos(3\pi x/L)[\cos(12\pi x/L)+1]$ {because $\cos^2(6\pi x/L)=[\cos(12\pi x/L)+1]/2$}

Thus we have:
$2\cos(3\pi x/L)\cos^2(6\pi x/L)=\cos(3\pi x/L)\cos(12\pi x/L)+\cos(3\pi x/L)=$
$[\cos(15\pi x/L)/2]+[\cos(9\pi x/L)/2]+\cos(3\pi x/L)$

Thus $F(x)=D+[A/2]\cos(15\pi x/L)+[A/2]\cos(9\pi x/L)+A\cos(3\pi x/L)$

As a result if we set t=0 in the solution for $u(x,t=0)=F(x)$ then weobtain that only the terms with n=0, 3, 9, 15 are non-zero so that

$a_0=D$
$a_3=A$
$a_9=A/2$
$a_{15}=A/2$
$a_n=0$ ($n\geq 1$ and $n\neq 3, 9, 15$)

Thus the solution for t>0 reads simply of the form:

$$u(x,t)=D+A\exp[-(3\pi xc/L)^2 t]\cos(3\pi x/L) + [A/2]\exp[-(9\pi xc/L)^2 t]\cos(9\pi x/L)$$
$$+[A/2]\exp[-(15\pi xc/L)^2 t]\cos(15\pi x/L)$$